TWO-DIMENSIONAL FLOW OF A PERFECT CONDUCTING GAS IN CROSSED ELECTRIC AND MAGNETIC FIELDS

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Reference [1, 2] give a solution of the problem of the two-dimensional flow of an inviscid thermally-nonconducting gas with constant conductivity in a channel of constant cross section for particular forms of the given applied magnetic field. The present paper obtains a solution of the problem of the two-dimensional flow of a gas with variable conductivity in crossed electric and arbitrary magnetic fields by means of the small parameter method. The magnetic Reynolds number R_m and the magnetohydrodynamic interaction parameter S are chosen as parameters. The international system of units is employed.

NOTATION

- V—flow velocity;
- j—electric current density;
- p—pressure in the flow;
- E—electric field strength;
- ρ—gas density;
- σ—electrical conductivity of the gas;
- T—gas temperature;
- $\varkappa$—ratio of specific heats at constant pressure and volume;
- L—channel half-height;
- μ—permeability (magnetic)
- B—magnetic induction vector;
- B_0—applied magnetic field;

1. We shall consider two-dimensional flow in the xy plane (Fig. 1.) of an inviscid* thermally-non conducting gas of variable conductivity in a channel of constant cross section in the presence of a constant electric field and an arbitrary magnetic field (applied in the xy plane). Moreover, we shall neglect the B_z component of the induced magnetic field. In this case

$$\mathbf{V}(u, v, 0), \quad \mathbf{B}(B_x, B_y, 0), \quad \mathbf{E}(0, 0, -E_0), \quad \frac{\partial}{\partial z} = 0. \tag{1.1}$$

Here E_0 = const for a channel with continuous electrodes.

We introduce the dimensionless quantities

$$u^\circ = \frac{u}{u_0}, \quad v^\circ = \frac{v}{u_0}, \quad p^\circ = \frac{p}{\rho_0 u_0^2}, \quad \rho^\circ = \frac{\rho}{\rho_0},$$

$$x^\circ = \frac{x}{L}, \quad y^\circ = \frac{y}{L}, \quad E^\circ = \frac{E_0}{u_0 B_{00}}, \quad B_x{}^\circ = \frac{B_x}{B_{00}},$$

$$B_y{}^\circ = \frac{B_y}{B_{00}}, \quad j^\circ = \frac{j}{\sigma_0 u_0 B_{00}}, \quad \sigma^\circ = \frac{\sigma}{\sigma_0}, \quad T^\circ = \frac{T}{T_0}. \tag{1.2}$$

In what follows only dimensionless quantities are employed and for simplicity the upper index ° is omitted.

In these variables the system of equations describing the flow under investigation has the form

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} - jB_y S \qquad \left(S = \frac{\sigma_0 L B_{00}^2}{\rho_0 u_0}\right)$$

$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + jB_x S, \qquad \frac{\partial(u\rho)}{\partial x} + \frac{\partial(v\rho)}{\partial y} = 0$$

$$\frac{1}{\varkappa-1}\left(u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y}\right) - \frac{\varkappa}{\varkappa-1}\frac{p}{\rho}\left(u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y}\right) = \frac{j^2 S}{\sigma}$$

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = R_m j \qquad (R_m = \mu\sigma_0 u_0 L) \tag{1.3}$$

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0, \quad j = \sigma(-E_0 + uB_y + vB_x), \quad \sigma = \sigma(p, T).$$

The boundary conditions for the gas-dynamical part of the problem have the form

$$u = 1, \quad \rho = 1, \quad v = 0, \quad p = p_0 \ \text{at} \ x = 0;$$

$$v = 0 \ \text{at} \ y = \pm 1 \tag{1.4}$$

We shall consider the case when S and R_m are small quantities. We shall seek a solution of system (1.3) in the form of a series*

$$z = z_{00} + Sz_1 + R_m z_2 + S^2 z_3 + SR_m z_4 + R_m{}^2 z_5 + \ldots \tag{1.5}$$

where z is understood to signify u, v, ρ, p, B_x, B_y, whence in view of (1.4)

$$u_{00} = 1, \rho_{00} = 1, v_{00} = 0, p_{00} = p_0, B_{x00} = B_{x0}, B_{y00} = B_{y0}.$$

On setting expressions (1.5) in the system of equations (1.3) and equating terms with like powers of S and R_m, respectively, we obtain a system of equations in the functions $u_1, v_1, p_1, \rho_1, u_2, v_2, p_2, \rho_2, B_{x1}, B_{y1}, B_{x2}, B_{y2}$. Thus for the first power in S we obtain

$$\frac{\partial u_1}{\partial x} = -\frac{\partial p_1}{\partial x} - \sigma(B_{y0} - E_0) B_{y0},$$

$$\frac{\partial v_1}{\partial x} = -\frac{\partial p_1}{\partial y} + \sigma(B_{y0} - E_0) B_{x0},$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial \rho_1}{\partial x} = 0,$$

$$\frac{1}{\varkappa-1}\left(\frac{\partial p_1}{\partial x} - \varkappa p_0 \frac{\partial \rho_1}{\partial x}\right) = \sigma(B_{y0} - E_0)^2, \tag{1.6}$$

*It is assumed that the dimension d is sufficiently large.

*Expansion in these parameters is also applied in [3] to the problem of the free flow of a compressible conducting fluid

$$\frac{\partial B_{y1}}{\partial x}-\frac{\partial B_{x1}}{\partial y}=0,\qquad \frac{\partial B_{x1}}{\partial x}+\frac{\partial B_{y1}}{\partial y}=0. \tag{1.7}$$

Similarly, for the first power in R_m we have

$$\frac{\partial u_2}{\partial x}=-\frac{\partial p_2}{\partial x},\qquad \frac{\partial v_2}{\partial x}=-\frac{\partial p_2}{\partial y},$$
$$\frac{\partial u_2}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial \rho_2}{\partial x}=0,\qquad \frac{\partial p_2}{\partial x}-\varkappa p_0\frac{\partial \rho_2}{\partial x}=0, \tag{1.8}$$

$$\frac{\partial B_{y2}}{\partial x}-\frac{\partial B_{x2}}{\partial y}=\sigma(B_{y0}-E_0),\qquad \frac{\partial B_{x2}}{\partial x}+\frac{\partial B_{y2}}{\partial y}=0. \tag{1.9}$$

It is clear from the equations that the functions u_1, v_1, p_1, ρ_1 depend on the applied magnetic and electric fields B_{x0}, B_{y0}, E_0. In general, the functions u_2, v_2, p_2, ρ_2, do not depend on B and E_0. It is possible to show, by examining system (1.3), that the equations in the system for determining the hydrodynamic functions and components of the magnetic induction vector separate in any approximation.

We obtain the boundary conditions for system (1.6) from (1.4)

$$u_1=0,\quad v_1=0,\quad p_1=0,\quad \rho_1=0\quad \text{at}\quad x=0,$$

$$v_1=0\quad \text{at}\quad y=\pm 1. \tag{1.10}$$

Here v_1 is a bounded quantity for $x\to\infty$.

2. To solve, we differentiate the first equation of (1.6) with respect to y, the second with respect to x and subtract the second result from the first. We obtain

$$\frac{\partial^2 u_1}{\partial x\,\partial y}-\frac{\partial^2 v_1}{\partial x^2}=$$
$$=-\frac{\partial}{\partial y}[\sigma B_{y0}(B_{y0}-E_0)]-\frac{\partial}{\partial x}[\sigma B_{x0}(B_{y0}-E_0)]. \tag{2.1}$$

With the help of the first three equations of (1.6) we eliminate the derivatives $\partial p_1/\partial x$ and $\partial\rho_1/\partial x$ from the fourth equation; we obtain the equation

$$-\frac{1}{\varkappa-1}\left[\frac{\partial u_1}{\partial x}+\sigma B_{y0}(B_{y0}-E_0)\right]+\left(\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}\right)\frac{p_0\varkappa}{\varkappa-1}=$$
$$=\sigma(B_{y0}-E_0)^2. \tag{2.2}$$

We differentiate (2.2) with respect to y and subtract the result from (2.1); we obtain

$$(1-M_0^2)\frac{\partial^2 v_1}{\partial x^2}+\frac{\partial^2 v_1}{\partial y^2}=q(x,y)$$
$$M_0^2=\frac{1}{\varkappa p_0},\qquad q(x,y)=(1-M_0^2)\frac{\partial}{\partial x}[\sigma B_{x0}(B_{y0}-E_0)]+$$
$$+\frac{\partial}{\partial y}\{\sigma(B_{y0}-E_0)[B_{y0}-(\varkappa-1)M_0^2(B_{y0}-E_0)]\}. \tag{2.3}$$

We shall consider the case of subsonic flow. Applying the method of G. A. Grinberg [4], we find for equations (2.3) with boundary conditions (1.10)

$$v_1=\sum_{k=1}^{\infty}\frac{1}{k\pi}\sin\frac{k\pi\eta}{2}\left\{\int_0^{\infty}q_k(z)\exp\left[-\frac{k\pi}{2}(z+\xi)\right]dz-\right.$$
$$-\int_0^{\xi}q_k(z)\exp\left[-\frac{k\pi}{2}(\xi-z)\right]dz-$$
$$\left.-\int_{\xi}^{\infty}q_k(z)\exp\left[-\frac{k\pi}{2}(z-\xi)\right]dz\right\},$$

$$q_k=\int_0^2 q\sin\frac{k\pi\eta}{2}d\eta,\qquad \xi=\frac{x}{\sqrt{1-M_0^2}},\qquad \eta=y+1. \tag{2.4}$$

For boundary conditions (1.10) we obtain

$$u_1=\frac{M_0^2}{1-M_0^2}\int_0^x\sigma(B_{y0}-E_0)[(\varkappa-1)(B_{y0}-E_0)+B_{y0}]\,dx-$$
$$-\frac{1}{1-M_0^2}\int_0^x\frac{\partial v_1}{\partial y}dx,\quad p_1=-\int_0^x\frac{\partial u_1}{\partial x}dx+\int_0^x\sigma B_{y0}(B_{y0}-E_0)\,dx,$$
$$\rho_1=-\int_0^x\left(\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}\right)dx, \tag{2.5}$$

from equations (2.2) and the first and third equations of (2.2) and the first and third equations of (1.6).

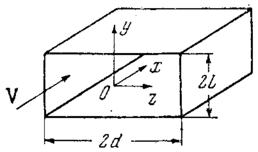

Fig. 1

We now go on to determine the functions u_2, v_2, ρ_2, p_2. A solution of equations (1.8) is constructed in a similar manner, and we obtain the equation

$$(1-M_0^2)\frac{\partial^2 v_2}{\partial x^2}+\frac{\partial^2 v_2}{\partial y^2}=0 \tag{2.6}$$

for the function v_2.

In accordance with (1.4) the boundary conditions for system (1.8) have the form

$$v_2=0,\quad u_2=0,\quad p_2=0,\quad \rho_2=0\quad \text{at}\quad x=0,$$

$$v_2=0\quad \text{at}\quad y=\pm 1, \tag{2.7}$$

while v_2 is a bounded quantity for $x\to\infty$.

With boundary conditions (2.7) system (1.8) has the null solution

$$v_2=0,\quad u_2=0,\quad p_2=0,\quad \rho_2=0. \tag{2.8}$$

If we investigate system (1.3) without confining ourselves to first-order powers in S and R_m in the expansions (1.5), then it is not difficult to show that all terms u_n, v_n, p_n appearing next to various powers of the parameter R_m are equal to zero.

Expansion (1.5) may then be represented as

$$z=z_{00}+S(z_1+R_m z_4+R_m^2 z_8+\ldots)+$$
$$+S^2(z_3+R_m z_7+R_m^2 z_{11}+\ldots)+\ldots,\quad (z=u,\,v,\,\rho,\,p).$$

For problems in which the dependence of the electrical conductivity of the gas on pressure and temperature cannot be neglected, the following method of successive approximations may be employed. We choose $\sigma = 1$ as a zeroth approximation and determine $p^{(0)}$, $\rho^{(0)}$ and $T^{(0)}$; the value of $\sigma(x, y)$ is found from the last formula of (1.3) and set in the right-hand sides of (2.4), (2.5). Subsequent approximations are calculated in a similar manner.

We have equations (1.7) and (1.9) for determining the functions B_{x1}, B_{y1}, B_{x2}. From the condition that the lines of force of the induced magnetic field be normal to the surface of the poles, which are parallel to the channel walls, we obtain the boundary conditions

$$B_{x1} = 0, \quad B_{x2} = 0 \quad \text{at} \quad y = \pm 1. \tag{2.9}$$

For a very long channel we may neglect the fringe effect at the inlet. We have as a result*

$$B_{x1} = B_{x2} = B_{y1} = B_{y2} = 0 \quad \text{at} \quad x = 0. \tag{2.10}$$

For conditions (2.9), (2.10) system (1.7) has the null solution

$$B_{x1} = 0, \quad B_{y1} = 0. \tag{2.11}$$

System (1.9) reduces to the equation

$$\frac{\partial^2 B_{x2}}{\partial x^2} + \frac{\partial^2 B_{x2}}{\partial y^2} = \frac{\partial}{\partial y}[\sigma(E_0 - B_{y0})]. \tag{2.12}$$

Using (2.9), (2.10), we obtain as before

$$B_{x2} = \sum_{k=1}^{\infty} \frac{1}{k\pi} \sin\frac{k\pi\eta}{2} \left\{ \int_0^{\infty} f_k(z) \exp\left[-\frac{k\pi}{2}(z + x)\right] dz - \right.$$

$$- \int_0^{x} f_k(z) \exp\left[-\frac{k\pi}{2}(x - z)\right] dz -$$

$$\left. - \int_x^{\infty} f_k(z) \exp\left[-\frac{k\pi}{2}(z - x)\right] dz \right\},$$

$$B_{y2} = \int_0^x \frac{\partial B_{x_2}}{\partial y} dx + \int_0^x \sigma(B_{y0} - E_0) dx,$$

$$f_k = \int_0^2 \frac{\partial}{\partial \eta}[\sigma(E_0 - B_{y0})] \sin\frac{k\pi\eta}{2} d\eta,$$

$$(\eta = y + 1). \tag{2.13}$$

It is not difficult to show that all terms B_{xn}, B_{yn} appearing as coefficients of different powers of the parameter S are equal to zero. Expansion (1.5) may then be represented in the following manner:

$$z = z_{00} + R_m(z_2 + Sz_4 + S^2 z_7 + \ldots) +$$

$$+ R_m^2(z_5 + Sz_8 + S^2 z_{12} + \ldots) + \ldots$$

$$(z = B_x, B_y).$$

The solution of systems of equations corresponding to higher powers of S and R_m is found in a similar way to the solution of systems (1.6)–(1.9).

3. The problem may be solved in a similar manner in the xz plane. In this case we take $v = 0$, $B_x = B_z = 0$, $\partial/\partial y = 0$, $\mathbf{E}(0, 0, -E_0)$. Here $E_0 = \text{const}$ for continuous electrodes. We shall consider that the dimension L is sufficiently large. This time we take d, the channel half-width, as the characteristic linear dimension.

The system of equations describing the flow in dimensionless form when the induced magnetic field is neglected is

$$\rho\left(u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \sigma B_y(E_0 - uB_y)S,$$

$$\rho\left(u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} - \sigma w B_y^2 S,$$

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial z}(\rho w) = 0. \tag{3.1}$$

$$\frac{1}{\varkappa - 1}\left[u\frac{\partial p}{\partial x} + w\frac{\partial p}{\partial z} - \frac{\varkappa p}{\rho}\left(u\frac{\partial \rho}{\partial x} + w\frac{\partial \rho}{\partial z}\right)\right] =$$

$$= \sigma[w^2 B_y^2 + (uB_y - E_0)^2]S,$$

$$u = 1, \quad w = 0, \quad \rho = 1, \quad p = p_0 \quad \text{at} \quad x = 0,$$

$$w = 0 \quad \text{at} \quad y = \pm 1. \tag{3.2}$$

We shall seek a solution of system (3.1) in the form

$$z = z_{00} + Sz_1 + S^2 z_3 + \ldots \quad (z = u, w, \rho p). \tag{3.3}$$

We note that in view of (3.2) $u_{00} = 1$, $w_{00} = 0$, $\rho_{00} = 1$, $p_{00} = p_0$.

We set (3.3) in equation (3.1). Making terms with like powers of S equal to zero, we obtain

$$\frac{\partial u_1}{\partial x} = -\frac{\partial p_1}{\partial x} + \sigma B_y(E_0 - B_y), \quad \frac{\partial w_1}{\partial x} = -\frac{\partial p_1}{\partial z},$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial w_1}{\partial z} + \frac{\partial \rho_1}{\partial x} = 0,$$

$$\frac{1}{\varkappa - 1}\left(\frac{\partial p_1}{\partial x} - \varkappa p_0 \frac{\partial \rho_1}{\partial x}\right) = \sigma(B_y - E_0)^2. \tag{3.4}$$

The solution of system (3.4) is obtained in a similar manner to that of system (1.6).

In conclusion, we note that, since we are investigating gas flow in a channel of constant cross section

*We must make use of the Biot-Savart formula if the boundary condition for B_{x1}, B_{y1}, B_{x2}, B_{y2} with $x = 0$ is to be more strictly formulated.

neglecting viscosity, two-dimensional flow can only result in the case when a two-dimensional magnetic field is applied. Thus, if we set

$$\sigma = \text{const}, B_{x0} = 0, B_{y0} = f(x) \tag{3.5}$$

for the first of the problems under consideration, then Eq. (2.4), (2.5) give the solution of the one-dimensional problem. Under these conditions we obtain from (2.4), (2.5)

$$v_1 = 0, \quad u_1 = Ax, \quad \rho_1 = -Ax, \quad p_1 = (c - A)x,$$

$$\left(A = \frac{B_0 - E_0}{p_0\varkappa - 1}[\varkappa(B_0 - E_0) + E_0], \quad c = B_0(E - B_0)\right) \tag{3.6}$$

for the particular case when $f(x) = B_0 = \text{const}$.

As opposed to known solutions of the one-dimensional problem for a channel of constant cross section [5–7], relations (3.6) have a very simple form.

Calculations carried out according to Eqs. (3.6) have shown that the departure from the exact solution [5] does not exceed 3%. In order to make this result more exact, we must employ the solution of systems corresponding to higher powers of the parameters S and R_m in the expansion (1.5).

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